MATHEMATICAL MODELING OF THE PROCESSES BEHIND THE FORMATION OF DETAILS

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A mathematical model can be served for the process of detail formation using the Hamilton–Ostrogradskii variation equation, which is written in the following form for the finite equilibrium state of an object [1]

$$\int_V [\mathbf{T} : \overline{\nabla}\delta\mathbf{U} + \mathrm{P}(\dot{\mathbf{V}} - \mathbf{K})\cdot\delta\mathbf{U}]\, dV = \int_\Omega \mathbf{T}_{\mathrm{N}}\cdot\delta\mathbf{U}\, d\Omega. \quad (1)$$

Here, T is the Cauchy stress tensor; T_N is the stress tensor on the surface Ω of the object with a unit normal N; $\dot{V}$, K are the vectors for the acceleration and the force; δU is the change in the displacement vector U; $\overline{\nabla}$ is the vector Hamiltonian operator; V, Ω are the volume and the surface of the object in four-dimensional space (including the time τ); and P is the density. The dots between the letters denote the scalar product of the tensor functions, and the dots above the letters represents the velocity. The capital letters pertain to functions which describe the finite state of the object.

To solve the problem, the variation equation (1) is transformed to the metric of some intermediate state which is, in general, in nonequilibrium. The transformation is done on the assumption that the functions which characterize the finite state are expressed in terms of a sum of the corresponding functions of the intermediate state (these functions will be written with lower-case letters according to the above definitions), and their increments are associated with the sign Δ. In addition, the transformation is facilitated by the condition for the conservation of mass and by the relation between the elementary surfaces of two configurations of a deformable object [2].

Hence, the transformation of the variation equation (1) leads to the following

$$\int_v [\mathbf{t} : \nabla\delta\mathbf{u} + \rho(\mathbf{v} - \mathbf{k})\cdot\delta\mathbf{u}]\, dv - \int_\omega \delta\mathbf{u}\cdot\mathbf{t}\cdot\mathbf{v}\, d\omega = -\int_v [\boldsymbol{\sigma} : \nabla\delta\mathbf{u} + \rho\Delta(\dot{\mathbf{v}} - \mathbf{k})\cdot\delta\mathbf{u}]\, dv + \int_\omega \delta\mathbf{u}\cdot\boldsymbol{\sigma}\cdot\mathbf{v}\, d\omega, \quad (2)$$

where

$$\boldsymbol{\sigma} = \Delta\mathbf{t} + (\mathbf{t} + \Delta\mathbf{t})\cdot\boldsymbol{\zeta};\quad \boldsymbol{\zeta} = \left(I^1_{\nabla\mathbf{u}} + I^2_{\nabla\mathbf{u}}\right)\mathbf{I} - \left(1 + I^1_{\nabla\mathbf{u}}\right)\nabla\mathbf{u} + (\nabla\mathbf{u})^2;$$

$I^1_{\nabla u}$, $I^2_{\nabla u}$ are the first and second invariants of the gradient of the displacement vector Δu; and I is the unit tensor.

The logarithmic Henke tensor [2] is used as a measure of the deformation

$$\mathbf{h} = \frac{1}{2}\ln(\mathbf{I} + \nabla\mathbf{u} + \mathbf{u}\nabla + \mathbf{u}\nabla\cdot\nabla\mathbf{u}), \quad (3)$$

which can be put into the form of a sum of the elastic h_e and plastic h_p components: $h = h_e + h_p$, which, because of the following equality

$$\mathbf{d} = \mathbf{h}^\nabla = \dot{\mathbf{h}} + \mathbf{h}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{h}$$

corresponds to the tensor for the deformation of the velocity $d = d_e + d_p$ (d is the velocity deformation tensor; w is the spin to which it corresponds; and h^∇ is the derivative of h with respect to Yaumann–Noll).

Relations can be written for the energy pair t − h (t − d) which arises from the associated law of plastic flow [3]

$$I^1_{\mathbf{h}} = \frac{1}{3K}\frac{\rho}{\mathrm{P}} I^1_{\mathbf{t}},\ I^1_{\mathbf{h}_p} = 0,\ \mathbf{h}'_e = \frac{1}{2\mu}\frac{\rho}{\mathrm{P}}\mathbf{t}',\ \mathbf{d}_p = \nu N\mathbf{a}. \quad (4)$$

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Here,

$$N=\left(\frac{\partial f}{\partial I_{\mathbf{h}}}I_{\mathbf{d}}^{1}+\frac{\partial f}{\partial \mathbf{h}_{e}'}:\mathbf{d}'\right)\left[\left(\frac{\partial f}{\partial \mathbf{h}_{e}'}-\frac{\partial f}{\partial \mathbf{h}_{p}}\right):\mathbf{a}\right]^{-1};$$

$$f=\frac{1}{2}\mathbf{a}:\mathbf{a}-\varkappa\leqslant 0;\quad \mathbf{a}=\frac{\rho}{\mathrm{P}}\mathbf{t}'-g\mathbf{h}_{p};$$

$$\nu=\begin{cases}1 & \text{for plastic deformation,}\\ 0 & \text{for elastic deformation;}\end{cases}$$

K, μ are elastic constants; $\kappa = \kappa(h_p)$ is the fluidity limit; $g = g(h_p)$ is an experimentally determined coefficient which characterizes the quantity of residual micro-stresses; f is the surface of the loading; a is the tensor for active stresses; and the apostrophes denote the deviated components of the tensor.

The formulation of the geometric (3) and physical (4) relations in the variation equation (2) allows one to transform it into an equation with a single unknown — the displacement vector u. This equation can be put in the form

$$\int_{v}[\mathbf{t}:\nabla\delta\mathbf{u}+\rho(\dot{\mathbf{v}}-\mathbf{k})\cdot\delta\mathbf{u}]\,dv-\int_{\omega}\mathbf{t}_{\nu}\cdot\delta\mathbf{u}d\omega=\int_{v}\nabla\mathbf{u}^{*}:{}^{(4)}\mathbf{m}:\nabla\delta\mathbf{u}dv+\int_{\omega}(\nabla\mathbf{u}:{}^{(4)}\mathbf{m}\cdot\mathbf{v})\cdot\delta\mathbf{u}d\omega, \tag{5}$$

where

$$\begin{aligned}{}^{(4)}\mathbf{m}=&\left(K-\frac{2}{3}\mu\right){}^{(4)}\mathbf{C}_{1}+2\mu{}^{(4)}\mathbf{C}_{3}+\left({}^{(4)}\mathbf{C}_{2}-{}^{(4)}\mathbf{C}_{3}\right)\cdot\mathbf{t}'+\left({}^{(4)}\mathbf{C}_{1}-{}^{(4)}\mathbf{C}_{3}\right)\cdot\mathbf{t}-2\mu\left\{\left[\left(2\mu+g+\frac{\partial g}{\partial\mathbf{h}_{p}}:\mathbf{h}_{p}\right)\mathbf{a}+\right.\right.\\&\left.\left.+\frac{\partial\varkappa}{\partial\mathbf{h}_{p}}\right]:\frac{\mathbf{a}}{2\mu}\right\}^{-1}\left({}^{(4)}\mathbf{C}_{3}-\frac{1}{3}{}^{(4)}\mathbf{C}_{1}\right):\mathbf{aa};\end{aligned}$$

${}^{(4)}C_i$ (i = 1, 2, 3) are the isotropic tensors of the fourth rank [2] which are functions only of the base vectors of the coordinate system.

Equation (5) is written in a form which is conveniently solved by an iteration technique. If the configuration of the deformable object corresponds to its equilibrium state, then the right-hand side of Eq. (5) goes to zero, and the solution of the problem is unknown. In the opposite case, the "discrepancy" of the solution allows one to determine the increment of the displacements over successive iterations and to correct the displacements, the configuration of the object, and other functions. For a converging process, the discrepancy (the right-hand side of the equation) goes to zero.

An additional difficulty in solving the problem of the deformation of an object in space which is restricted by a rigid matrix involves the formation of an algorithm for the exit of the object at the surface of the matrix and for the continued passage of the object along this surface taking into account friction.

The moment at which the boundary points of the object and the surface of the matrix intersect is given by the equation

$$f(x_1, x_2, x_3)=0 \tag{6}$$

in the space x_k (k = 1, 2, 3) and is determined over each n-th step of loading from the associated solution of the equation for the vector r for the position of the node points of the object's boundary

$$\mathbf{r}_{(n)}=\mathbf{r}_{(n-1)}+\mathbf{u}_{(n)} \tag{7}$$

and of Eq. (6).

If the boundary point of the object intersects the matrix or slides along its surface, then additional conditions regarding the displacements must be applied on the displacement of the node. In this case, reactive forces will act on the boundary of the object which support it on the surface of the matrix. The reaction is along the normal to the surface of the matrix, and the friction coefficient is determined as the force of friction (which is the corresponding tangential stress τ) according to the Amonton—Kulon law or to other considerations [4].

The force which acts on the boundary points of the object due to the matrix along the tangent is in the same direction as the force of friction. In the first approximation, this force is determined from the results of solving the problem when there are no displacements of the boundary points. Hence, one can determine the tangential contact stress t^{α}. One then has two situations

$$t^{\alpha} < \tau \text{ and } t^{\alpha} \geqslant \tau .$$

In the first case, which corresponds to "adhesion" of the points of the object to the surface of the matrix, the displacements of the object's boundary points are equal to zero, and in the second case, one considers the frictional forces determined by the quantity τ, and the displacements in the direction which is tangent to the surface of the matrix are found from the solution of the problem.

Writing Eq. (5) in the symbols of direct tensor calculus facilitates reducing it to matrix form, which corresponds to the method of finite elements. One must choose an isoparametric finite element for which the fundamental displacements that describe the configuration of the object and the displacement field are approximated by single functions. For formalization of the transfer from the tensor equation (5) to a description in coordinate-matrix form, the summation over all nodes of the finite element model for a deformable object is done according to the indices in the parentheses, and the summation over the coordinates is done using indices without parentheses.

One can use the above rule to put the terms of Eq. (5) in the form of a sum. For example,

$$\int_{v} \nabla \mathbf{u} : {}^{(4)}\mathbf{m} : \nabla \delta \mathbf{u}\, dv = \sum_{(j)} \delta W^{q}_{(j)} \sum_{(i)} W^{p}_{(i)} \int_{v} (\tilde{\nabla}_{k} \Phi^{(i)})^{n}_{pk} m^{ks}_{nl} (\tilde{\nabla}_{s} \Phi^{(j)})^{l}_{qs}\, dv.$$

Here, $W^{q}_{(j)}$ is the nodal displacement of the j-th node in the direction x^{q}; $\Phi^{(i)}$ is an approximation function for the i-th node; $m^{ks}_{n\ell}$ are the physical components of the tensor ${}^{(4)}m$ in the indicated coordinate system; and $\tilde{\nabla}_{k}$ is an operator related to the covariant derivative of the tensor. The other terms in expression (5) are given in a similar manner.

If one sets the factor equal to zero when the variations are independent, one can reduce the variation equation (5) to a system of matrix equations, which corresponds to the method of finite elements.

One uses a stepwise method of loading with an internal iteration cycle for solving the problem. Hence, the solution is accurate because, first, the iteraction process converges at the loading step and, second, the quantity of the step is elected by using a computer. The latter is necessary only when using the differential theory of plasticity and when analyzing the behavior of the object when its boundary moves along the surface of the matrix, i.e., in situations where the accuracy of the calculations is a function only of the accuracy of tracing through loading. For free deformations of an object made of a nonlinearly elastic material, the accuracy of the calculations for converging iteration processes does not depend on the size of the loading step.

It was found on a computer, using the law of plastic flow and assuming an error in the solution of no more than 1%, that the loading step, which is illustrated in Fig. 1, must be selected so that the increment of the maximum displacements of the points of a deformable shell is no greater than 0.1 of its thickness. The loading step was elected by extrapolating the curve for the dependence of the loading on the warping.

The above algorithm was used to compose a program for a computer using the language PL/1. This program can be used to detail the technological processes for processing metals with static loading pressures and to determine hardness, rigidity, and stability.

Some examples of solutions to the problem are given below which illustrates the potential of the program. A solution was obtained for deformation of axially symmetric shells. This was done because of the lack of machine time (for the stretching of the shell illustrated in Fig. 1, five hours of machine time was consumed on an ES-1040 computer) and because of the lack of experimental data on the behavior of materials for complex loading. Loading functions were put into the computer which are described by the Mizes surface with translational and isotropic reinforcement.

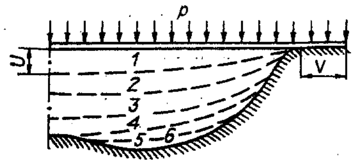

Fig. 1

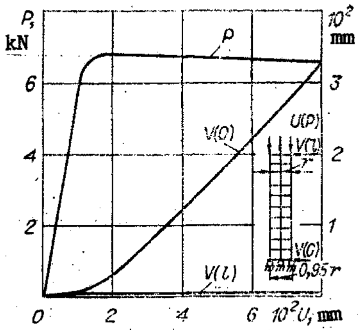

Fig. 2

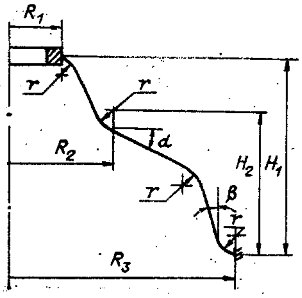

Fig. 3

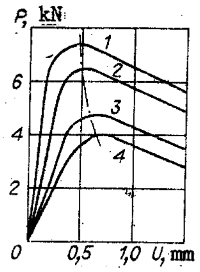

Fig. 4

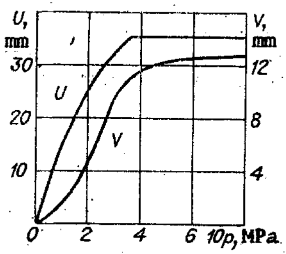

Fig. 5

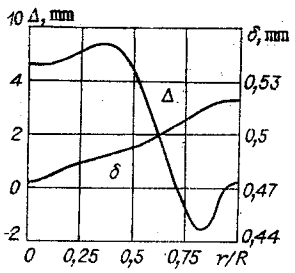

Fig. 6

For an analysis of the formation of a neck in a protonated circular sample with a length of ℓ = 200 mm and a radius of r = 10 mm (Fig. 2), the dimensions of the finite elements in the radial direction, which are joined to the center of the sample at the surface, were reduced by 5% in comparison to the base dimension (r(0) = 9.5 mm).

The finite element model for the fourth sample (taking into account symmetry) was formed by breaking the sample down into 10 equal layers along the axis and two layers along the radius. It was assumed that the sample was made of an ideal elastoplastic material with $E = 2 \cdot 10^5$ MPa, t_p = 220 MPa.

The sample was loaded on its end face by a displacement of U along the axis. It is evident from Fig. 2 that the displacement of the surface points of the center layer of the sample in the radial direction V(0) increases beginning from some time (the neck grows), while the surface points which are distinct from the center cross-section V(ℓ) are not displayed in the radial direction. In the imperfect region of the sample, one observes localization of the plastic deformations. When the force P attains its maximum value the sample loses stability during stretching.

When solving the problem of the deformation of a shell with a complex form (Fig. 3) made of soft steel with a thickness of 0.5 mm and dimensions of β = 22°, R_1 = 49 mm, R_2 = 52 mm, R_3 = 67 mm, r = 1 mm, H_1 = 21 mm, H_2 = 15 mm, one must choose an angle α for one of the sections of the shell such that one achieves buckling when the shell is compressed by a specific amount. The diagram for the deformation of the material was put into the computer on a pointwise basis and was then approximated by spline functions of the third order.

TABLE 1

Number of position	$10p$, MPa	V, mm	Number of steps	Number of position	$10p$, MPa	V, mm	Number of steps
1	0,5	0,6	100	4	2,75	9,25	369
2	1,33	2,55	200	5	3,5	11,25	410
3	2,225	5,93	300	6	6,6	11,35	460

The determination of the critical force is obtained by assigning the displacement of a rigid ring U mounted on the opening of the shell. One then solves a series of problems for the shell using various α to derive a dependence for the force P(U). One can use approximations to determine the force as a function of α (Fig. 4, α = 35, 32, 27, 22° correspond to lines 1-4).

The intermediate (dashed lines) and final forms of the details are shown in Fig. 1 which were obtained by extreme stretching of circular sheet stock made of the material AMG-6M. The quantity of loading steps and the values for the horizontal displacement as functions of the pressure p are shown in the table. The figures in the first column correspond to the positions of the shell indicated in Fig. 1. The change in some of the functions which characterize the behavior of the shell in the process of formation is indicated in Figs. 5 and 6. The horizontal displacement of the edge of the detail V and the vertical displacement of the center of the shell U as functions of p are shown in Fig. 5, and a change in the thickness δ and the spring back Δ along the length of the meridian of the shell r/R are shown in Fig. 6. According to the data from calculations and from the experiment, the shell was obtained after a single transition. The behavior of the change in the spring back has been confirmed experimentally.

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